Illustration - 51 *Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated.*

(A) 288 **(B)** 144 **(C)** 72 **(D)** 576

SOLUTION : (A)

Considering 4 particular flowers as one group of flowers, we have five flowers (one group of flowers and remaining four flowers) which can be strung to form a garland in 4!/2 ways. But 4 particular flowers can be arranged amongst themselves in 4! ways. Thus, the required number of ways = $\frac{4! \times 4!}{2} = 288$

DIVISION OF IDENTICAL OBJECTS INTO GROUPS — Section - 8

8.1 Introduction

In this section, we will discuss how to find number of ways to divide identical objects into groups.

For example, if we have to divide 5 identical copies of a book among 3 boys such that each boy gets at least 1 copy, then it can be achieved as shown below :

Boy 1	Boy 2	Boy 3
3	1	1
1	3	1
1	1	3
2	2	1
1	2	2
2	1	2

i.e. 5 identical copies can be divided in 6 ways.

We can study following formulae to find number of ways to divide them instead of writing ways and counting them.

8.2 Formulae

(a) The number of ways to divide n identical objects into r groups (different) such that each group gets 0 or more objects (empty groups are allowed) = ${}^{n+r-1}C_{r-1}$.

Proof :

Let $x_1, x_2, x_3, \ldots, x_r$ be the number of objects given to groups 1, 2, 3, . . . , r respectively.

As total objects to be divided is n, we can take

Sum of the objects given to all groups = n

$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 + \ldots + x_r = n.$

This equation is known as integral equation as all variables are integer.

As each group can get 0 or more, following are constraints on integer variables.

$0 \le x_1 \le n$; $0 \le x_2 \le n, \ldots,$ $\quad 0 \le x_r \le n$ $\quad$ *i.e.* $\quad 0 \le x_i \le n$ $\quad i = 1, 2, 3, \ldots, r.$

We can observe that number of integral solutions of the above equation is equal to number of ways to divide n identical objects among r groups such that each gets 0 or more.

How to find number of solutions ?

To find number of solutions, following method can be used. The derivation of this method is out of scope of JEE preparations.

Consider r brackets corresponding to r groups. In each bracket, take an expression given by $x^0 + x + x^2 + \ldots + x^n$. Here the various powers of x, i.e. 0, 1, 2, . . . , n correspond to the number of objects each group can take in the division.

Since the total number of objects is n. So, the required number of ways is the coefficient of x^n in the product

$$\underbrace{(x^0 + x + x^2 + \ldots + x^n)(x^0 + x^1 + \ldots + x^n) \ldots (x^0 + x^1 + x^2 + \ldots + x^n)}_{(r \text{ brackets})}$$

Thus, the required number of ways

$$= \text{Coefficient of } x^n \text{ in } (x^0 + x^1 + x^2 \ldots + x^n)^r$$

$$= \text{Coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^r = \text{Coefficient of } x^n \text{ in } (1 - x^{n+1})^r\ (1-x)^{-r}$$

$$= \text{Coefficient of } x^n \text{ in } (1-x)^{-r} \qquad [\because\ x^{n+1} \text{ cannot be used to generate } x^n \text{ term}].$$

$$= {}^{n+r-1}C_n = {}^{n+r-1}C_{r-1}.$$

[Using : coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$ and ${}^nC_r = {}^nC_{n-r}$].

(b) The number of ways to divide n identical objects into r groups (different) such that each group receives at least one object (empty groups are not allowed).

$$= {}^{n-1}C_{r-1}.$$

Let $x_1, x_2, x_3, \ldots, x_r$ be the number of objects given to groups 1, 2, 3, . . . r respectively.

As total objects to be divided is n, we can take sum of the objects given to all groups $= n$

$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 + \ldots + x_r = n \qquad$ **. . . (i)**

This is an integral equation as all variables are integer. Each group should get at least one object.

$\Rightarrow \quad 1 \le x_1 \le n\ ;\ 1 \le x_2 \le n, \ldots, \quad 1 \le x_r \le n \quad$ *i.e.* $\quad 1 \le x_i \le n \quad i = 1, 2, 3, 4, \ldots, r.$

We can observe that number of solutions of above equation is equal to number of ways to divide n objects among r groups such that each group gets at least one.

Proof :

The number of solutions of integral equation (i)

$$= \text{coeff. of } x^n \text{ in } (x + x^2 + \ldots + x^n)^r$$

Now, Coeff. of x^n in $(x + x^2 + x^3 + \ldots + x^n)^r$

$$= \text{Coeff. of } x^n \text{ in } x^r(1 + x^2 + \ldots\ x^{n-1})^r = \text{Coeff. of } x^{n-r} \text{ in } (1 + x + x^2 + \ldots x^{n-1})^r$$

$$= \text{Coeff. of } x^{n-r} \text{ in } \left(\frac{1-x^n}{1-x}\right)^r = \text{Coeff. of } x^{n-r} \text{ in } (1-x)^{-r} \quad [\text{using: } x^n \text{ cannot be used to generate } x^{n-r}]$$

$$= {}^{n-r+r-1}C_{n-r} = {}^{n-1}C_{n-r} = {}^{n-1}C_{r-1}$$

(c) The number of ways to divide n identical objects in r groups (different) such that each group gets minimum m objects and maximum k objects

$= \text{Coefficient of } x^n \text{ in } (x^m + x^{m+1} + \ldots\ldots + x^k)^r$

The logic of the above can be understood after reading proofs of (a) and (b) results.

Illustration - 52 *Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.*

(A) 12 **(B)** 6 **(C)** 24 **(D)** 72

SOLUTION : (B)

Let x_1, x_2 and x_3 be the number of balls into three boxes so that no box is empty and each box being large enough to accomodate all the balls.

The number of ways of distributing 5 balls into Boxes 1, 2 and 3 is the number of integral solutions of the equation $x_1 + x_2 + x_3 = 5$ subjected to the following conditions on x_1, x_2, x_3. . . . (i)

Conditions on x_1, x_2 and x_3 :

According to the condition that the boxes should contain at least one ball, we can find the range of x_1, x_2 and x_3 i.e.

$\text{Min}(x_i) = 1$ and $\text{Max}(x_i) = 3$ for $i = 1, 2, 3$ [using : $\text{Max}(x_1) = 5 - \text{Min}(x_2) - \text{Min}(x_3)$]

or $1 \le x_i \le 3$ for $i = 1, 2, 3$

So, number of ways of distributing balls

= number of integral solutions of (i)

$= \text{coeff. of } x^5 \text{ in the expansion of } (x + x^2 + x^3)^3$

$= \text{coeff. of } x^5 \text{ in } x^3(1 - x^3)(1 - x)^{-3} = \text{coeff. of } x^2 \text{ in } (1 - x^3)(1 - x)^{-3}$

$= \text{coeff. of } x^2 \text{ in } (1 - x)^{-3}$ [as x^3 cannot generate x^2 terms]

$= {}^{3+2-1}C_2 = {}^4C_2 = 6$

Another Approach :

The number of ways to divide n identical objects into r groups such that no group remains empty

$= {}^{n-1}C_{r-1}$ [using result 8.2 (b)]

$= {}^{5-1}C_{3-1} = {}^4C_2 = 6$

Illustration - 53 *How many integral solutions are there to $x + y + z + t = 29$, when $x \ge 1$, $y \ge 2$, $z \ge 3$ and $t \ge 0$?*

(A) 5200 **(B)** 2600 **(C)** 1300 **(D)** 650

SOLUTION : (B)

We have,

$x \ge 1, y \ge 2, z \ge 3$ and $t \ge 0$, where x, y, z, t are integers

Let $u = x - 1,\ v = y - 2,\ w = z - 3.$

Then, $x \ge 1 \Rightarrow u \ge 0$; $y \ge 2 \Rightarrow u \ge 0$; $z \ge 3 \Rightarrow w \ge 0$

Thus, we have

$u + 1 + v + 2 + w + 3 + t = 29 \Rightarrow u + v + w + t = 23$ [where $u \ge 0$; $v \ge 0$; $w \ge 0$]

The number of solutions of above equation is equal to number of ways to divide 23 identical objects among 4 groups such that each gets 0 or more.

$\Rightarrow$ The total number of solutions of this equation is

$= {}^{23+4-1}C_{4-1} = {}^{26}C_3 = 2600$ [Using result given in 8.2]

Illustration - 54 *Find the number of ways of distributing 10 identical balls in 3 boxes so that no box contains more than four balls and less than 2 balls ?*

(A) 6 **(B)** 12 **(C)** 24 **(D)** 36

SOLUTION : (A)

Let x_1, x_2 and x_3 be the number of balls placed in Boxes 1, 2 and 3 respectively.

Number of ways of distributing 10 balls in 3 boxes

= Number of integral solutions of the equation $x_1 + x_2 + x_3 = 10$ . . . (i)

Conditions on x_1, x_2 and x_3

As the boxes should contain atmost 4 ball and at least 2 balls, we can make

Max $(x_i) = 4$ and Min $(x_i) = 2$ for $i = 1, 2, 3$

or $2 \le x_i \le 4$ for $i = 1, 2, 3$

So the number of ways of distributing balls in boxes

= number of integral solutions of equation (i)

= coeff of x^{10} in the expansion of $(x^2 + x^3 + x^4)^3$

= coeff of x^{10} in $x^6 (1 - x^3)^3 (1 - x)^{-3}$ = coeff of x^4 in $(1 - x^3)^3 (1 - x)^{-3}$

= coeff of x^4 in $(1 - {}^3C_1 x^3 + {}^3C_2 x^6 + \dots)(1 - x)^{-3}$

= coeff of x^4 in $(1 - x)^{-3}$ – coeff of x in ${}^3C_1 (1 - x)^{-3}$

$= {}^{4+3-1}C_4 - 3 \times {}^{3+1-1}C_1 = {}^6C_4 - 3 \times {}^3C_1 = 15 - 9 = 6$

Illustration - 55 *The number of ways in which 30 marks can be alloted to 8 questions if each question carries atleast 2 marks,is :*

(A) ${}^{21}C_7$ **(B)** ${}^{21}C_8$ **(C)** ${}^{13}C_7$ **(D)** *None of these*

SOLUTION : (A)

Let, the marks given in each question be ;

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ [where x_i's ≥ 2 $(i = 1, 2, \dots, 8)$]

and $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 30$. Let $x_i = y_i + 2$ $\forall$ $i = 1, 2, 3 \dots\dots, 8$.

$\therefore$ $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = 30 - 16$ $\Rightarrow$ $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = 14$

where $y_i \ge 0$ $\forall$ $i = 1, 2, 3, \dots, 8$

$\Rightarrow$ Number of solutions of the above equation

= Number of ways to divide 14 identical objects among 8 groups such that each group gets 0 or more.

$\Rightarrow$ Number of solutions $= {}^{14+8-1}C_{8-1} = {}^{21}C_7$ [Using result given in 8.2]